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A multidimensional superposition principle: classical solitons II

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Abstract
A concept introduced previously as an approach for finding superposition formulae for solutions of nonlinear PDEs and an explanation of various types of wave interactions in such systems is developed further, both from the theoretical and technical point of view. In its framework, which is the framework of the multidimensional superposition principle, a straightforward and self-consistent technique for constructing the related invariant manifolds in a soliton case is proposed. The method is illustrated by simple examples, which, in particular, show in principle the generality that exists between superposition formulae for conventional linear and nonlinear soliton equations. The demonstration that the so-called truncated singular expansions associated can be with some sort of the above soliton invariant manifolds is also presented.

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1. Introduction

Although the concept proposed in [1] for finding superposition laws for solutions of PDEs imposes no restrictions on their type, the advantages of that approach become most obvious in the case of soliton equations, and perhaps its most significant feature is the explanation of the soliton interaction mechanism without involving the inverse scattering transform (IST) [2].

In the same paper [1] a structure of *general solutions* describing interactions of a soliton with other, *arbitrary*, perturbations was shown practically for nonlinear PDEs to which the truncated singular expansions technique [3, 4] is applicable. Although the majority of known solitonic models integrable by the IST are found among the last ones, there also exist 'nonintegrable' soliton systems interesting for applications, see e.g. [5, 6]. From this point of view, developing a general technique in the framework of the multidimensional superposition approach is very important. This is the goal of the present work.

2. The multidimensional superposition principle and invariant manifolds of the soliton type

2.1. The definition of invariant manifolds of the soliton type

Suppose we have some PDE, linear or nonlinear, for the sake of definiteness, of evolution type in 1D, and for simplicity not depending explicitly on the independent variables

$$\frac{\partial}{\partial t}u = E\left(\frac{\partial}{\partial x}; u\right) \qquad u = u(x,t) \tag{1}$$

with the function $u(x,t)$ being *the projection*

$$u(x,t) = u(x_1, x_2, t)|_{x_1=x_2=x} \tag{2}$$

of another function, where the original spatial variable x is split. In doing so, the latter has to satisfy the following equation:

$$\frac{\partial}{\partial t}u = E\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}; u\right) \qquad u = u(x_1, x_2, t) \tag{3}$$

that will be called the *d-adjoint* to equation (1). Let this last equation for $u(x_1, x_2, t)$ in its turn have an *invariant manifold* [7] such that among all the differential relations describing it there is one of the form

$$Q(u, u_{x_1}, \ldots, u_{kx_1}) = 0 \qquad k \in N. \tag{4}$$

Respectively, the remaining relations (if any) will be of the form

$$G_i\left(\frac{\partial}{\partial x_2}; u, u_{x_1}, \ldots, u_{(k-1)x_1}\right) = 0 \qquad i = \overline{1, m}; m \in N \tag{5}$$

taking into account the elimination of terms with derivatives $\frac{\partial}{\partial t}$ and $\frac{\partial^k}{\partial x^k}$ in view of (3), (4). Here and further (e.g., in proposition 1) we will suppose without loss of generality that all the necessary equations (here (3) and (4)) are formally resolvable with respect to such leading derivatives. Moreover, as will be seen from the examples, when an initial NPDE is a polynomial, it presents no technical problem.

In the general case, invariant manifolds just narrow the set of possible solutions. Introducing here equation (4) in the form of an ODE (this is the only limitation), we demand *separation* of the variables x_1 and x_2 (in the generalized sense analogous to [8]). In our original space this corresponds to splitting of the solution $u(x_1, x_2, t)$ into components that can be described independently of each other. Moreover, one of them associated with x_1 appears to be fixed, in contrast to the component depending on x_2. In fact, (4) fully determines the dependence with respect to x_1, and simultaneously introduces the new functions $\varphi_j(x_2, t)$, $j = \overline{0, k-1}$ as parameters. The remaining equations, equations (5), in their turn determine a linkage between the φ_j. As a result, the corresponding solutions (2) collapse into the independent (spatially) ingredients associated with the different independent variables. In doing so, the general structure of u or *a superposition formula* for them is fixed and uniquely determined by (4), (5). Such a paradigm was called *a multidimensional superposition principle* [1]. The fact that here there exists some fixed and therefore stable component associated with (4) may be interpreted as the presence of a soliton in a solution, and the solution $u(x_1, x_2, t)$

itself can be interpreted as the soliton envelope with the parameters φ_j modulated by a perturbation. Because of this it is logical to call equation (4) a *soliton envelope equation*, and invariant manifolds of the above form *invariant manifolds of the soliton type*.

Note 1. All the aforesaid results are immediately generalized both to cases of any dimension and to systems of equations. In so doing, splitting can be performed for all or only for part of the independent variables, as required for a concrete investigation.

Note 2. Equations explicitly depending on the independent variables, e.g.,

$$\frac{\partial}{\partial t}u = E\left(x; \frac{\partial}{\partial x}, \frac{\partial}{\partial t}; u\right) \qquad u = u(x, t) \tag{6}$$

are considered in the same way. In the general case such a problem is, as usual, reduced to the previous one for an appropriate system by means of the formal introduction of an auxiliary dependent variable, so that finally we obtain the complete system

$$\frac{\partial}{\partial t}u = E\left(X; \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}; u\right) \qquad u = u(x_1, x_2, t)$$
$$X_t = 0 \qquad X_{x_1} + X_{x_2} = 1 \qquad X = X(x_1, x_2)$$

instead of one equation. It is possible to investigate particular cases with some concrete dependence of the d-adjoint equation with respect to x_1, x_2

$$\frac{\partial}{\partial t}u = E\left(x_1, x_2; \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}; u\right) \qquad u = u(x_1, x_2, t) \tag{7}$$

compatible with the corresponding projection of (7) to (6).

Note 3. Analogously, there is no need to introduce the independent variables t, x_1, x_2 into (4), (5) explicitly. A system

$$\frac{\partial}{\partial t}u = E\left(\frac{\partial}{\partial x}; u\right) \qquad u = u(x, t)$$
$$X_t = 0 \qquad X_x = 1 \qquad X = X(x)$$
$$T_t = 1 \qquad T_x = 0 \qquad T = T(t)$$

with the above type IMS can be considered instead.

It is necessary to emphasize one circumstance. Separation of the variables x_1, x_2 and the existence of a SF for the related components do not yet signify their physical separation in our observable x-space as well as the possibility of their separate existence. However, if this takes place before and after a soliton–perturbation interaction, when in the space there is some domain where the solution $u(x_1, x_2, t)$ depends only on x_1 (the 'soliton' variable) and another domain where it depends already only on x_2 (the 'perturbation' variable), the values of soliton parameters (the asymptotical values of φ_j) may be different, because they are determined by perturbation asymptotes. In other words, in the general case a soliton before and after an interaction is in different states. A standard phase shift or changes of the kink velocities and wave numbers [6] are examples of such *switching from one state to another*.

The following examples, in spite of their triviality, well illustrate the essence of the multidimensional superposition.

Example 1. Consider the conventional linear heat equation

$$u_t - u_{xx} = 0 \qquad u = u(x, t) \tag{8}$$

and its d-adjoint for the function $u(x_1, x_2, t)$

$$u_t - u_{x_1x_1} - 2u_{x_1x_2} - u_{x_2x_2} = 0 \tag{9}$$

It is easy to see that the latter, equation (9), exhibits the existence of the following IMS:

$$\begin{aligned} Q &= u_{x_1x_1} - ku_{x_1} = 0 \qquad k = \text{const} \\ G &= u_{x_1x_2} = 0 \end{aligned} \tag{10}$$

By this means, as a result, from (9) and (10) we have the SF

$$u = \varphi_1\, \mathrm{e}^{kx_1+k^2t} + \varphi_0(x_2, t) \qquad \varphi_1 = \text{const}$$

where the 'free' function $\varphi_0(x_2, t)$ has to satisfy an equation of the original form (8). Since $\varphi_0(x_2, t)$ can, in particular, be equal to zero, the component $\varphi_1 \exp(kx_1 + k^2t)$ is also a solution of the original equation.

Obviously, the above IMS or similar ones are suitable for any linear PDE in $(1+1)$ D with constant coefficients and just means that the Fourier mode can be added to an arbitrary solution. Also, it is clear that for linear PDEs other IMSs corresponding to various superpositions can be constructed.

Example 2. The equation

$$u_t - \left(1 + \frac{u_x}{u}\right)^2 = 0 \qquad u = u(x, t)$$

is presumably the simplest nonlinearizable soliton equation. Its d-adjoint analogue

$$u_t - \left(1 + \frac{u_{x_1} + u_{x_2}}{u}\right)^2 = 0 \qquad u = u(x_1, x_2, t) \tag{11}$$

has the IMS

$$Q = u_{x_1x_1}u - 2u_{x_1}^2 - uu_{x_1} = 0 \qquad G = u_{x_1x_2}u - 2u_{x_1}u_{x_2} = 0$$

which results in the following SF:

$$u = \frac{1}{\varphi_1\, \mathrm{e}^{x_1} + \varphi_0^{-1}(x_2, t)} \qquad \varphi_1 = \text{const} \tag{12}$$

with $\varphi_0(x_2, t)$ being a solution of the original form of the equation.

Here both a perturbation and the soliton part can exist separately. If $\varphi_{0x_2}(\infty, t) = 0$ or, according to (11)

$$\varphi_0(\pm\infty, t) = t + \theta_{\pm\infty} \qquad \theta_{\pm\infty} = \text{const}$$

solutions (12) are reduced to the expression

$$u_{\text{kink}} = \frac{1}{\varphi_1\, \mathrm{e}^{x_1} + (t + \theta_{\pm\infty})^{-1}}$$

corresponding to a kink with a time-dependent velocity and amplitude. Simultaneously, we also have

$$u(+\infty, x_2, t) = 0 \qquad u(-\infty, x_2, t) = \varphi_0(x_2, t)$$

In other words, (12) may describe an interaction of the above kink with a localized perturbation.

In these examples both solutions in the SFs satisfy the initial equations. But as is demonstrated below, such a situation may not be true in all cases.

Example 3. Let us consider the IMS

$$\begin{aligned} Q &= -u_{x_1x_1}u + 2u_{x_1}^2 + kuu_{x_1} = 0 \qquad k = \text{const} \\ G &= -u_{x_1x_2}u + 2u_{x_1x_2} - kuu_{x_1} = 0 \end{aligned} \tag{13}$$

for the equation

$$u_t = u^2 + \left(\frac{u_{x_1} + u_{x_2}}{u}\right)^2 \qquad u = u(x_1, x_2, t) \tag{14}$$

d-adjoint to

$$u_t = u^2 + \left(\frac{u_x}{u}\right)^2 \qquad u = u(x, t).$$

When $k = 0$, we have from (13) and (14) the following SF:

$$u = \frac{1}{\varphi_1 x_1 - \left(\varphi_1^2 + 1\right)t + \varphi_0(x_2, t)} \qquad \varphi_1 = \text{const}.$$

And this solution describes an interaction of the pole solution

$$u = \frac{1}{\varphi_1 x_1 - \left(\varphi_1^2 + 1\right)t} \tag{15}$$

of the initial equation with another component that, however, is governed by

$$\varphi_{0_{t_2}} + (2\varphi_1)\varphi_{0_{x_2}} + \varphi_{0_{x_2}}^2 = 0 \tag{16}$$

that cannot be linked with (14) directly. This means that a perturbation can be observable only during an interaction (in the domain when $u_{x_1} \neq 0$) and cannot exist separately, so that

$$u(\infty, x_2, t) = 0.$$

It is not superfluous to underline here that a perturbation and the pole solution themselves are independent of each other. Moreover, if one looks at (15) and the linearized version of (16), it is seen that the pole and small enough perturbations move with different velocities, $\varphi_1 + \varphi_1^{-1}$ and $2\varphi_1$, respectively.

The case with $k \neq 0$ leads to the SF

$$u = \frac{1}{\varphi_1 \,\mathrm{e}^{k(x_1 - x_2)} + \varphi_0^{-1}(x_2, t)} \qquad \varphi_1 = \text{const} \tag{17}$$

where only the function φ_0 satisfies the initial equation. This SF has altogether a different sense. In fact, we will obtain after projection (2)

$$u(x, t) = \frac{1}{\varphi_1 + \varphi_0^{-1}(x, t)}$$

so that (17) corresponds to the trivial one-parameter transformation.

2.2. *Finding IMSs for equations with a polynomial nonlinearity*

Now let us discuss some questions associated with finding the invariant manifolds (4), (5) for equations (3) with a polynomial nonlinearity.

At present the formal theory of overdetermined differential systems such as (3)–(5) with the above type nonlinearity has been developed adequately by itself. There exist a number of approaches, and their computer implementations allowing one to work with such systems: to prove their compatibility or, conversely, incompatibility, to bring them to some given form, in particular to the involutory form and so on. Equations (3)–(5) are a typical example of such a system, the compatible overdetermined system of NPDEs. The problem is that our goal here is, namely, to determine the corresponding form of Q and G_i. Hence, for its solution we should in turn *first derive the determining equations to Q and G_i* considering them as unknown functions and their arguments as independent variables and then, after that, work with these equations using, e.g., specialized computer algebra packages such as `CRACK` [9], `RifSimp` [10] or `DiffGrob2` [11]. As a result, the existing theory and methods have to be applied twice, both when constructing the above-mentioned equations to Q, G_i and when solving them. Unfortunately, the programs existing for these purposes can be used only at the second stage.

Our purpose here is not, and cannot be, to give a detailed description of an algorithm for deriving the equations for Q, G_i and determining their possible form, although such an algorithm reduces to a finite number of cross-differentiations, excluding some derivatives from one equation by means of others, and so on. A description of such algorithms and the associated theory (see, e.g., [12–14]) is a separate, special field of contemporary mathematics, on one hand, it is beyond the scope of one or several papers and, on the other, exhaustively elucidated in the related literature. However, some aspects associated with our specific problem and the main principles need to be presented here. In so doing, knowledge of only basic notions (such as the weight or ranking of variables, etc) is needed, all of which cannot be presented in this work, but are well explained and are accessible, in particular, in the manuals of the above-mentioned computer packages.

Below, the main principles and steps for finding IMSs, which are imagined to be most optimal now in view of the form of (3)–(5), are presented.

In section 2.1 the form of IMSs has been indicated in quite broad outline. It has just been said about the existence of two types of equations in an IMS: a soliton envelope equation and linkage equations. In doing so, the latter can be brought to various forms, and instead of some initial set of equations (5) we can use any equivalent set obtained by combination of them and their differential consequences. Because of this it is necessary to chose some concrete form for them that would allow us to effectively work with such equations and clearly formulate their properties. For this, first of all, it is necessary to introduce a suitable derivative ranking. This is the `LEX` ranking, such that a differentiation with respect to t has the highest weight, and its order will be taken into account first, then analogously a differentiation on x_2 and finally a differentiation on x_1. By this means, in our case the following ordering will take place:

$$u_t \succ u_{\alpha x_2, k x_1} \succ u_{\alpha x_2,(k-1)x_1} \succ \cdots \succ u_{\alpha x_2} \succ u_{(\alpha-1)x_2,kx_1} \succ \cdots \succ u_{x_1} \succ u \tag{18}$$

where α is the maximal available order of differentiation on x_2 (for x_1 this is obviously k). With regard to the introduced ranking the following form of IMSs for a d-adjoint equation:

$$u_t = E\left[u, \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)u, \ldots, \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^n u\right] \qquad u = u(x_1, x_2, t) \qquad n \in N \tag{19}$$

will be called *canonical* and used when finding IMSs:

(1) A soliton envelope equation is assumed to be resolved with respect to its leading derivative

$$u_{kx_1} = q\big(u_{(k-1)x_1}, u_{(k-2)x_1}, \ldots, u_{x_1}, u\big) \qquad k \in N \tag{20}$$

(2) Linkage equations (if any)

$$G_i\big(u_{\alpha_i x_2, i x_1}, \ldots, u\big) = 0 \qquad \alpha_i \in N \quad i = \overline{l, k-1} \quad 0 \leqslant l \leqslant k-1 \tag{21}$$

($u_{\alpha_i x_2, i x_1}$ corresponds to a leading (see (18)) derivative from the available ones in an equation) cannot be simplified further by means of each other and equation (20) and contain all their differential consequences on x_1.

Proposition 1. *For equations (21) the relation*

$$\alpha_l \geqslant \alpha_{l-1} \geqslant \cdots \geqslant \alpha_{k-1} \tag{22}$$

takes place.

Proof. This follows directly from the irreducibility requirement for equations (21) under the above ranking. Indeed, assume that there exist some integers f and h $(l \leqslant f < h \leqslant k-1)$ such that $\alpha_f < \alpha_h$. Then differentiating the related equation $G_f = 0$ $(h - f)$ times with respect to x_1, one obtains the equation with the leading derivative $u_{\alpha_f x_2, hx_1}$. But, as a result, the equation $G_h = 0$ can be simplified, because its leading derivative $u_{\alpha_h x_2, hx_1}$ can be eliminated by using the latter. □

Proposition 2. *In the system (19)–(21) equation (20) and all of equations (21) with $\alpha_j \leqslant n_1$, where n_1 is some integer, themselves constitute a compatible subsystem.*

Proof. Since the compatibility conditions $G_{j_{x_1}} = 0$ $(j = \overline{l_1, k-1}, l \leqslant l_1)$ of such a subsystem are part of the related conditions for the whole system (19)–(21) and do not contain the derivatives $u_{j_1 x_2, j_2 x_1}$ with $j_1 > n_1$, then they must be satisfied without regard for the remaining equations (21) and, of course, without equation (19). □

Proposition 3. *The number l corresponds to the number of 'free' function parameters in a solution envelope.*

Proof. Since the subsystem (21) contains all its differential consequences of x_1, we can consider it as the system of the $(k-l)$ differential only on x_2 equations to k functions $v_i = u_{ix_1}$ $(i = \overline{0, k-1})$. By this means l functions v_i remain free, corresponding to the availability of l arbitrary, in this subsystem framework, functions of x_2 (and t, of course) in the expression for u. □

Note 4. Usually (soliton equations) $l = 1$, i.e. there exists the only 'free' function parameter corresponding to arbitrariness in perturbation chosen. However, the cases $l > 1$ and $l = 0$ are also possible. The latter means that the related SF describes superposition of a soliton only with some specific perturbations. It does not mean, however, that superposition with arbitrary perturbations is impossible, because for this it may be necessary to consider a higher order IMS (i.e. assuming a higher number of modulated parameters in a soliton envelope). For our purposes we will further suppose that $l \neq 1$.

Proposition 4. *For the greatest value α_l in (22) the relation*

$$\alpha_{\max} = \alpha_l \leqslant n(k-l)$$

is true.

Proof. Indeed, the compatibility condition for (19) and (20) $u_{t,kx_1} - u_{kx_1,t} = 0$ is an expression depending on the derivatives $u_{j_1 x_2, j_2 x_1}$ with $j_1 \leqslant n_1 \leqslant n$ (in a nondegenerate case—when it is not satisfied identically). Respectively, this demands the availability of the equations $G_{i_1} = 0$ with $\alpha_{i_1} \leqslant n_1$ in (21). These equations are compatible with (20) (see proposition 2) but optionally with (19). In the last case equations with $\alpha_{i_2} \leqslant n_2 \leqslant n_1 n \leqslant n$ have to be subject to (21) in order to satisfy the above-mentioned compatibility conditions $\frac{\partial}{\partial t} G_{i_1} = 0$. Analogously, if these new equations $G_{i_2} = 0$ are incompatible with (19) directly, the presence of the equations with $\alpha_{i_2} \leqslant n_2 n \leqslant 3n$ is necessary, and so on. Since the quantity of equations (21) is limited (proposition 3), we arrive at the above estimation. □

Next, on the strength of the form of equation (19), when finding IMSs, it is more convenient to work with the operators $\frac{\partial}{\partial x_1}$ and $D_x = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ instead of $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$. (Further, for the latter we will again use the familiar subscript notation

$$\begin{aligned} u_x &\equiv D_x u = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) u \\ u_{ix} &\equiv D_x^i u = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^i u \qquad i \in N. \end{aligned} \tag{23}$$

It is necessary to remember, however, that by this the composite differential operator is meant rather than simply derivatives.) Transformation of equations (19)–(21) from one form to another is trivial and unique (with regard to (20)). It is essential here that under the above ranking (18) and its D_x-operator version

$$u_t \succ u_{\alpha x, kx_1} \succ u_{\alpha x,(k-1)x_1} \succ \cdots \succ u_{\alpha x} \succ u_{(\alpha-1)x,kx_1} \succ \cdots \succ u_{x_1} \succ u$$

the transformation

$$\begin{aligned} u_{j_1 x, j_2 x_1} = u_{j_1 x_2, j_2 x_1} + R\left(u_{(j_1-1)x_2,(k-1)x_1}, \ldots, u_{(j_1-1)x_2}; \ldots; u_{(k-1)x_1}, \ldots, u\right) \qquad j_1 \in N \\ 0 \leqslant j_2 \leqslant k-1 \end{aligned} \tag{24}$$

and the reverse transformation

$$u_{j_1 x_2, j_2 x_1} = u_{j_1 x, j_2 x_1} - \tilde{R}\left(u_{(j_1-1)x,(k-1)x_1}, \ldots, u_{(j_1-1)x}; \ldots; u_{(k-1)x_1}, \ldots, u\right)$$

do not change leading terms in G_i. In other words, when finding IMSs, *instead of equations* (19)–(21) *we can work with the D_x-presentation*

$$u_t = E(u, \ldots, u_{nx}) \qquad u = u(x_1, x_2, t) \qquad n \in N \tag{25}$$

$$u_{kx_1} = q\left(u_{(k-1)x_1}, u_{(k-2)x_1}, \ldots, u_{x_1}, u\right) \qquad k \in N \tag{26}$$

$$G_i\left(u_{\alpha_i x, i x_1}, \ldots, u\right) = 0 \qquad \alpha_i \in N \quad i = \overline{l, k-1} \quad 0 \leqslant l \leqslant k-1 \tag{27}$$

that are considerably more compact and simple.

We now consider concrete schemes for constructing IMSs (already in the presentation (25)–(27)) and first of all a *direct approach*.

Step 1. Consider a system initially consisting of two equations only, namely, equation (25) with a known right-hand side, d-adjoint to an equation of interest, and a soliton envelope equation (26) with some predetermined k (here already the form of the right-hand side q is unknown). Calculate the compatibility condition

$$u_{t,kx_1} - u_{kx_1,t} = 0 \tag{28}$$

for them. This expression is a polynomial with respect to the kernels $u_{j_1 x, j_2 x_1}$ ($j_1 = \overline{1, n}$; $j_2 = \overline{0, k-1}$) or, in view of (24), with respect to derivatives $u_{j_1 x_2, j_2 x_1}$ correspondingly. (The dependence on the derivatives $u_{j_2 x_1}$ ($j_2 = \overline{0, k-1}$) is unknown because it is associated with q.)

The next step depends on what supposition is chosen about the essence of (28). Two variants are possible:

(a) There exists a function q such that (28) becomes the identity $0 \equiv 0$, i.e. any equations $G_i = 0$ setting linkages between the parameters are absent, and we want to find its form.
(b) The equation obtained is satisfied for some function q with regard to *all or possibly only some of* the linkage equations (27) in the IMS sought.

Step 2(a). In this case we have to obtain a system of equations for the function q for further simplification and solving, or to prove its incompatibility for any q. To obtain such a system it is necessary and sufficient to equate to zero coefficients at the different monomials composed from the kernels u_{j_1x,j_2x_1} ($j_2 \neq 0$), because in view of (24) this is equivalent to equating to zero coefficients at the related monomials with $u_{j_1x_2,j_2x_1}$.

Step 2(b). The expression for (28) written through the derivatives of q has form (27) by itself, and our goal here is to bring it together with (26) to form a compatible system. Differentiating it with respect to x_1 the necessary number of times and simplifying the resulting expressions, one finds a set of relations of type (27) again. They can be separated into the linkage equations and the conditions of their compatibility with (26) (the determining equations for q). Since the first ones are optionally compatible with (25), the further construction and consideration of the related equations are also necessary in turn. Here again two cases/steps are possible.

Step 3(a). The obtained set of relations (27) is final, and therefore only the fulfilment of the compatibility conditions with (25) is necessary. The last ones are obtained by differentiating (27) with respect to t and after simplification these are added to the equation determining q.

Step 3(b). The part of the compatibility conditions (see step 3(a)) is added to the relations of type (27) obtained previously. After simplification they together make the new candidate for (27) in the sought for IMS. The remaining relations are added to the determining equations of q.

Further steps analogous to *2(b)*, *3(b)* are repeated till construction of the system (25)–(27) with the necessary number of equations (27) is complete (note 4). The process is finished at a step of type *2(a)* or *3(a)*. Simultaneously we obtain the set of compatibility conditions.

The next two examples with the second-order soliton envelope equation

$$u_{x_1x_1} = q(u, u_{x_1}) \tag{29}$$

illustrate the ideas. Here and further for conciseness we will use the notation

$$u_{ix,jx_1} = u_{ij} \qquad u_{ix} = u_i \qquad i \geqslant 0 \quad j > 1. \tag{30}$$

Example 4. Take the d-adjoint

$$u_t = u_{xx} \qquad u = u(x_1, x_2, t)$$

of the conventional heat equation. Its compatibility condition with (29) gives

$$u_{tx_1x_1} - u_{x_1x_1t} = 2u_{11}u_1q_{uu_{01}} + u_1^2q_{2u} + u_{11}^2q_{2u_{01}} = 0. \tag{31}$$

Ignore the case without linkages (step 2(a)). Then here the only variant possible is

$$u_{xx_1} = s(u, u_x, u_{x_1}) \tag{32}$$

with s being the solution of (31) for u_{xx_1}. Since there can be only one linkage equation, (29), (32) are the final IMS. The compatibility conditions

$$u_{x_1x_1x} - u_{xx_1x_1} = u_1q_u + sq_{u_{01}} - u_{01}s_u - qs_{u_{01}} - ss_{u_1} = 0$$
$$u_{txx_1} - u_{xx_1t} = 2u_1ss_{uu_{01}} + 2u_1u_2s_{uu_{01}} + u_1^2s_{2u} + 2u_2ss_{u_1u_{01}} + s^2s_{2u_{01}} + u_2^2s_{2u_1} = 0$$

together with (31) with respect to s make the system for q, s and are the determining equations for the IMS. Solving them, e.g., by `RifSimp`, leads to expressions (10) from example 1.

Example 5. Slightly change the equation introducing the nonlinearity to

$$u_t = uu_{xx} \qquad u = u(x_1, x_2, t).$$

Again its compatibility condition with (29) is

$$2uu_1u_{11}q_{uu_{01}} + uu_1^2 q_{2u} + uu_{11}^2 q_{2u_{01}} - u_{01}u_2 q_{u_{01}} + u_2 q + 2u_{01}u_{21} = 0.$$

And the first possible case for the linkage equation is obviously

$$u_{xxx_1} = s\big(u, u_x, u_{x_1}, u_{xx_1}, u_{xx}\big). \tag{33}$$

The other compatibility conditions will be

$$\begin{aligned} u_{x_1x_1xx} - u_{xxx_1x_1} &= 2u_1u_{11}q_{uu_{01}} + u_1^2 q_{2u} - u_1 q_u s_{u_{11}} + u_2 q_u + u_{11}^2 q_{2u_{01}} \\ &\quad - u_{11}s_{u_{11}}q_{u_{01}} + s q_{u_{01}} - u_{01}s_u - q s_{u_{01}} - u_{11}s_{u_1} - s s_{u_2} = 0 \\ u_{txxx_1} - u_{xxx_1t} &= E_1([s], u, u_1, u_{01}, u_{11}, u_2, u_3, u_4) = 0 \text{ (26 terms totally).} \end{aligned} \tag{34}$$

We can, however, continue the procedure, interpreting (34) as the new linkage equation involving (33), so that

$$u_{xx_1} = s\big(u, u_x, u_{x_1}, u_{xx}\big). \tag{35}$$

The related IMS will consist of equations (29), (35) with the determining equations

$$\begin{aligned} u_{txx_1} - u_{xx_1t} &= 2uu_1q_{uu_{01}}s + uu_1^2 q_{2u} + uq_{2u_{01}}s^2 - u_{01}u_2q_{u_{01}} + 2u_{01}u_1s_u \\ &\quad + 2u_{01}ss_{u_{01}} + 2u_{01}u_2s_{u_1} + 2u_{01}u_3s_{u_2} + u_2q = 0 \\ u_{x_1x_1x} - u_{xx_1x_1} &= u_1q_u + q_{u_{01}}s - u_1s_us_{u_2} - u_{01}s_u - s_{u_{01}}s_{u_2}s - s_{u_{01}}q - u_2s_{u_1}s_{u_2} \\ &\quad - s_{u_1}s - u_3s_{u_2}^2 = 0 \\ u_{txx_1} - u_{xx_1t} &= E_2([s], u, u_1, u_{01}, u_2, u_3) = 0 \text{ (17 terms totally)} \end{aligned}$$

to q and s.

In contrast to the previous example, here the determining systems for both IMSs are inconsistent.

The direct scheme considered is complicated for realization, when n or $k > 2$. The difficulty is that although every step is reduced to the standard operations for work with overdetermined systems, simultaneously they demand a practically interactive mode when considering a large number of branches. An *indirect approach* may be more effective. In this case we consistently consider the variants of G_i with all possible leading derivatives, without loss of generality, simultaneously assuming their resolution with respect to the last ones, i.e. we look over all the systems

$$\begin{aligned} &u_t = E(u, \ldots, u_{nx}) \qquad u = u(x_1, x_2, t) \qquad n \in N \\ &u_{kx_1} = q\big(u_{(k-1)x_1}, u_{(k-2)x_1}, \ldots, u_{x_1}, u\big) \qquad k \in N \end{aligned} \tag{36}$$

$$u_{\alpha_i x, ix_1} = g_i\big(u_{\alpha_i x,(i-1)x_1}, \ldots, u\big) = 0 \qquad \alpha_i \in N \quad i = \overline{l, k-1} \quad 0 \leqslant l \leqslant k-1 \tag{37}$$

The compatibility conditions for them are

$$u_{t,kx_1} - u_{kx_1,t} = 0 \qquad u_{t,\alpha_i x, ix_1} - u_{\alpha_i x, ix_1,t} = 0 \qquad u_{kx_1,\alpha_i x} - u_{\alpha_i x, kx_1} = 0$$

to make the determining equations for q and g_i directly suitable for automatic simplification and solving. The consequence of this is the necessity to deal with all cases assumable by proposition 4, while a number of them may be incompatible with a concrete form of (36) (see, e.g., examples 4 and 7). Such 'empty' branches, however, are comparatively quickly detected and discarded during calculations. For small n and k a mixed strategy is effective as well. It uses both steps from the direct scheme and the substitutions (37) which are reasonable at each step.

2.3. Multisoliton formulae of superposition. A universal form for IMSs collective and abstract variable differential operators

As a result of the procedures described above, in the case when the determining equations have a solution, we obtain system (25)–(27) written in terms of the operator $D_x = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$. Taking into account (23), this system is brought to the form (19)–(21) already with respect to the differential operators on x_1 and x_2 convenient for deriving a SF or an investigation of such. Inversely, form (25)–(27) can be obtained from (19)–(21) by the formal replacement $u_{x_2} = D_x u - u_{x_1}$ and this in turn *plays no less of an important role*.

First of all, since a perturbation in a SF is arbitrary (the case $l \neq 0$), it can again include in itself a soliton component or components, so that the variable x_2 can be split further, say as

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2'} + \cdots + \frac{\partial}{\partial x_m'} + \frac{\partial}{\partial x_{m+1}'} \qquad m \geqslant 2$$

where x_i' $(i = \overline{2, m})$ are associated with the solitons in this perturbation. On the other hand, x_1 can, in principle, be associated with any of the available solitons, and for each of them one can write respectively (the prime is discarded hereafter)

$$\begin{aligned} &u_{kx_j} = q\big(u_{(k-1)x_j}, u_{(k-2)x_j}, \ldots, u_{x_j}, u\big) \qquad j = \overline{1, m} \\ &G_i\big(u_{\alpha_i x, ix_j}, \ldots, u\big) = 0 \qquad i = \overline{l, k-1} \\ &u = u(x_1, \ldots; x_{m+1}; t) \end{aligned} \tag{38}$$

Here the subscript 'x', e.g., u_x, already denotes the action of the new, 'extended', operator D_x

$$D_x = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_m} + \frac{\partial}{\partial x_{m+1}}.$$

By this means, a construction of an IMS for 'm solitons plus a perturbation' reduces to simple multiplication of relations already found for the case of 'one soliton plus a perturbation' and further verification of compatibility for this new set of equations. Of course, in the same way one can couple IMSs corresponding to different types of solitons (for instance, bell-shaped and kink-shaped ones).

Note 5. It is necessary to remember that systems constructed in this manner may appear to be incompatible, that is multisoliton solutions may simply not exist, e.g., in the case of kinks with different asymptotes.

Next, the same form equations such as (26), (27) with regard to the change $x_1 \longrightarrow t_1$

$$\begin{aligned} &u_{kt_1} = q\big(u_{(k-1)t_1}, u_{(k-2)t_1}, \ldots, u_{t_1}, u\big) \\ &G_i\big(u_{\alpha_i x, it_1}, \ldots, u\big) = 0 \qquad i = \overline{l, k-1} \qquad u = u(x, t_1, t_2) \end{aligned} \tag{39}$$

will take place, if the t variable is split as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}$$

and then considered instead of x (or together with x).

Finally, the equations

$$\begin{aligned} &u_{k\tau} = q\big(u_{(k-1)\tau}, u_{(k-2)\tau}, \ldots, u_\tau, u\big) \\ &G_i\big(u_{\alpha_i x, i\tau}, \ldots, u\big) = 0 \qquad i = \overline{l, k-1} \qquad u = u(x, t, \tau) \end{aligned} \tag{40}$$

obtained from (26), (27) by the formal change $\frac{\partial}{\partial x_1} \longrightarrow \frac{\partial}{\partial \tau}$ ($D_x \equiv \frac{\partial}{\partial x}$ in this case) also have the sense and will be compatible with (25) and each other. Such a subsystem determines the dependence of u on some free parameter τ (see note 6).

In all three cases (the splitting of x or/and t with the related 'soliton' envelope equation, the case of a free parameter) IMSs (38)–(40) associated with an initial equation

$$u_t = E(u, \ldots, u_{nx}) \qquad n \in N \tag{41}$$

can be written in the unified form

$$u_{kz} = q\big(u_{(k-1)z}, u_{(k-2)z}, \ldots, u_z, u\big) \tag{42}$$

$$G_i\big(u_{\alpha_i x, iz}, \ldots, u\big) = 0 \qquad i = \overline{l, k-1}, k, l \in N \tag{43}$$

Here the notation u_t, u_x and so on corresponds to differentiation with respect to all the related split coordinates

$$u_{i_1 t} \equiv D_t^{i_1} u = \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_{j_1}}\right)^{i_1} u \qquad u_{i_2 x} \equiv D_x^{i_2} u = \left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_{j_2}}\right)^{i_2} u$$

and the operator $\frac{\partial}{\partial z}$ (u_z and so on) corresponds to any of the following real differential operators:

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial t_i} \qquad u = u(x_1, \ldots, x_{j_2}; t_1, \ldots, t_{j_1}) \qquad 1 \leqslant i \leqslant j_1 \quad j_1 \in N \tag{44}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x_j} \qquad u = u(x_1, \ldots, x_{j_2}; t_1, \ldots, t_{j_1}) \qquad 1 \leqslant j \leqslant j_2 \quad j_2 \in N \tag{45}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau} \qquad u = u(x, t, \tau) \tag{46}$$

(t_i, x_j are some of the 'soliton' variables). Because of this, it is reasonable to call the unified form (41)–(43) *the universal form for IMSs* and the operators D_x (D_t) and $\frac{\partial}{\partial z}$ respectively *the collective and abstract variable differentiation operators*. (In doing so, however, it is important to remember that the concrete coordinate equations (42), (43) will be of a quite different sense, and will lead to different expressions for the function u.)

Note 6. In the case (46), $\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau}$, equations (43) can be considered as special type of generalization of nonclassical symmetries. In fact, equations (41) and (43) themselves, without (42), make the compatible subsystem. In doing so, they determine the dependence of the function $u(x, t, \tau)$ on the parameter τ, analogous to the equation

$$u_\tau = \sigma(u, u_x, u_{xx}, \ldots)$$

with σ being a classical symmetry [15]. (The latter is the particular case of (43).) While (42) means that this dependence is fixed.

Such symmetries are associated with the invariance of IMSs when (44) or (45) with respect to the trivial transformations $x_j \longrightarrow \tilde{x}_j + \tau$ or $t_i \longrightarrow \tilde{t}_i + \tau$, and the related invariant solutions [15] correspond to solutions $u(x, t)$ without a soliton component.

We stress, separately, that the above-mentioned IMSs and symmetries are directly linked to one another, but that an IMS involves in itself an additional requirement—the splitting of some *concrete variable*.

3. Some examples of reaction–diffusion equations with the simplest IMSs

Below, the technique outlined in the previous section is illustrated by means of several examples. In doing so, some other aspects, in particular the possibility of the classification of equations, will be considered as well.

Example 6. Consider the following problem. Assume that we have the general form second-order evolution equation or, more precisely, already its d-adjoint, see (23), version

$$u_t = f(u, u_x, u_{xx}) \qquad f_{u_{xx}} \neq 0 \tag{47}$$

and we wish to determine all permissible f in (47) and q in the related soliton envelope equation

$$u_{x_1x_1} = q(u, u_{x_1}) \tag{48}$$

such that any linkage equations in the IMS are absent. In other words, their compatibility condition has to be satisfied identically. The last one is the polynomial with respect to u_{xxx_1}, namely

$$u_{tx_1x_1} - u_{x_1x_1t} = B_2 u_{xxx_1}^2 + B_1 u_{xxx_1} + B_0 = 0$$

where the coefficients B_i $(i = \overline{0, 2})$ are written as

$$\begin{aligned}
B_2 &= f_{2u_2}\\
B_1 &= u_{01} f_{uu_2} + u_{11} f_{u_1u_2}\\
B_0 &= 2u_{01}u_{11} f_{uu_1} + u_{01}^2 f_{2u} - u_{01} f_u q_{u_{01}} + f_u q + u_{11}^2 f_{2u_1} + u_1 f_{u_1} q_u + 2u_1 u_{11} f_{u_2} q_{uu_{01}}\\
&\qquad + u_1^2 f_{u_2} q_{2u} + u_2 f_{u_2} q_u + u_{11}^2 f_{u_2} q_{2u_{01}} - f q_u
\end{aligned}$$

(Here the shortened notation of (30) has again been used.) Since, obviously, the B_i do not depend on u_{2xx_1} $(u_{2x_2x_1})$, we can equate them to zero, i.e. $B_i = 0$, and one has from $B_2 = 0$

$$f(u, u_1, u_2) = f_0(u, u_1) + f_1(u, u_1)u_2 \qquad f_1 \neq 0 \tag{49}$$

In the same manner, after substituting (49) into equations $B_1 = 0$ and $B_0 = 0$, we should equate to zero the coefficients of the different powers of u_2 and then u_{11}, so that finally we arrive at the system

$$\begin{aligned}
&f_{1u_1} = 0 \qquad f_{1u} = 0 \qquad u_{01} f_{12u} - u_{01} f_{1u} q_{u_{01}} + f_{1u} q + u_1 f_{1u_1} q_u = 0\\
&f_{02u_1} + f_1 q_{2u_{01}} = 0 \qquad u_{01} f_{0uu_1} + u_1 q_{uu_{01}} f_1 = 0\\
&u_{01}^2 f_{02u} - u_{01} f_{0u} q_{u_{01}} + f_{0u} q + u_1 f_{0u_1} q_u + u_1^2 f_1 q_{2u} - f_0 q_u = 0.
\end{aligned} \tag{50}$$

As seen, f_1 is a constant

$$f_1(u, u_1) = c_1 = \text{const} \neq 0 \tag{51}$$

and system (50) is simplified further

$$\begin{aligned}
&f_{02u_1} + c_1 q_{2u_{01}} = 0\\
&u_{01} f_{0uu_1} + c_1 u_1 q_{uu_{01}} = 0\\
&u_{01}^2 f_{02u} - u_{01} f_{0u} q_{u_{01}} + f_{0u} q + u_1 f_{0u_1} q_u + c_1 u_1^2 q_{2u} - f_0 q_u = 0.
\end{aligned} \tag{52}$$

Equation (52) leads to an additional separation of the variables, and we must set

$$f_0(u, u_1) = f_{00}(u) + f_{01}(u)u_1 + f_{02}(u)u_1^2 \tag{53}$$

$$q(u, u_{01}) = q_0(u) + q_1(u)u_{01} + q_2(u)u_{01}^2 \tag{54}$$

with

$$f_{02}(u) = -c_2 q_2(u) \qquad c_2 = \text{const} \tag{55}$$

After that it is possible to equate to zero the coefficients of the powers of u_1 and then u_{01} in the remaining equations. Thus one has the relations

$$f_{01}(u) = c_2 \tag{56}$$

$$q_1(u) = c_3 \qquad c_3 = \text{const} \tag{57}$$

together with the equations

$$q_{02u} - q_{0u}q_2 - q_{2u}q_0 = 0 \tag{58}$$

$$f_{002u} - f_{00u}q_2 - f_{00}q_{2u} = 0 \tag{59}$$

$$f_{00u}q_0 - f_{00}q_{0u} = 0 \tag{60}$$

and, as a result, taking into account (49), (51), (53)–(57), one obtains the final form of f in (47) and q in (48)

$$u_t = f_{00}(u) + c_2u_x - c_1q_2(u)u_x^2 + c_1u_{xx} \qquad c_1 \neq 0 \tag{61}$$

$$u_{x_1x_1} = q_0(u) + c_3u_{x_1} + q_2(u)u_{x_1}^2 \tag{62}$$

with the additional linkages (58)–(60) between $q_0(u)$, $q_2(u)$ and $f_{00}(u)$. It is easy to verify that the last relations correspond to the fact that (61) and (62) can be obtained from the related second-order linear equations by the simple point transformation

$$u \longrightarrow g(u) : g_{uu} + q_2(u)g_u = 0.$$

This result is not accidental. It can be shown that in the general case the absence of linkage equations indicates linearization of an equation by means of a point transformation. This example also demonstrates that we can work with equations of a general form for the purpose of their classification.

Example 7. For the next example we try to find the IMSs associated again with the second-order soliton envelope equation

$$u_{x_1x_1} = q\big(u, u_{x_1}\big) \tag{63}$$

for the following equation:

$$u_t = f(u)u_x + u_{xx} \qquad f_u \neq 0 \tag{64}$$

and construct the related SFs.

In the first step, calculation of the compatibility condition for (63) and (64) brings us to the expression

$$u_{11}^2q_{2u_{01}} + 2u_{11}\big(u_{01}f_u + u_1q_{uu_{01}}\big) + u_1^2q_{uu} + u_1\big(u_{01}^2f_{2u} - u_{01}f_uq_{u_{01}} + f_uq\big) = 0. \tag{65}$$

As a consequence, the only type of linkage equation possible is

$$u_{xx_1} = g\big(u, u_{x_1}, u_x\big). \tag{66}$$

(According to proposition 4, one more case $u_{xxx_1} = g\big(u, u_{x_1}, u_x\big)$ could be assumed.)

In the next step, we have to calculate the compatibility conditions between (66) and (63) and between (66) and (64). The resulting equations are as follows:

$$\begin{aligned} u_2^2g_{2u_1} + u_2\left(u_{01}f_u + 2u_1g_{uu_1} + 2g_{u_{01}u_1}g\right) + u_{01}u_1^2f_{2u} - u_{01}u_1f_ug_{u_{01}} - u_1^2f_ug_{u1} \\ + 2u_1f_ug + 2u_1g_{uu_{01}}g + u_1^2g_{2u} + g_{2u_{01}}g^2 = 0 \end{aligned} \tag{67}$$

$$u_1q_u + q_{u_{01}}g - u_{01}g_u - g_{u_{01}}q - g_{u_1}g = 0 \tag{68}$$

for $u_{txx_1} - u_{xx_1t} = 0$ and $u_{x_1x_1x} - u_{xx_1x_1} = 0$, respectively.

In principle, the system (65), (67), (68) is already suitable for processing by specialized packages such as `CRACK` [9] or `RifSimp` [10]. But here it can be further simplified. We see that the coefficients at the powers of u_2 in (67) have to be equal to zero, $g_{2u_1} = 0$, i.e.

$$g(u, u_{01}, u_1) = g_0(u, u_{01}) + g_1(u, u_{01})u_1. \tag{69}$$

Taking into account the last expression (66) for g, `RifSimp`, e.g., gives the following simplified system for the unknown functions $g_0(u, u_{01})$, $g_1(u, u_{01})$, $q(u, u_{01})$ and $f(u)$:

$$g_{1u} = -\frac{g_1^2}{u_{01}} \qquad g_{1u_{01}} = \frac{g_1}{u_{01}} \qquad g_{0u} = \frac{g_0 g_1}{u_{01}} \qquad g_{0u_{01}} = \frac{g_0}{u_{01}}$$
$$q_u = -2g_1^2 \qquad q_{u_{01}} = \frac{2u_{01}g_1 + q}{u_{01}} \qquad f_u = -2\frac{g_1 g_0}{u_{01}^2}$$

which is easily integrated

$$q(u, u_{01}) = \frac{2u_{01}^2}{u + c_1} + \lambda u_{01} \tag{70}$$

$$g_0(u, u_{01}) = -\frac{c_2(u + c_1)u_{01}}{2} \tag{71}$$

$$g_1(u, u_{01}) = \frac{u_{01}}{u + c_1} \tag{72}$$

$$f(u) = c_2 u + c_3 \qquad c_1, c_2, c_3, \lambda = \text{const} \tag{73}$$

As seen from (73), there exists the only equation of the type (64) with the second-order soliton envelope equation, namely the well-known Burgers' equation.

Taking into account expressions (70)–(73), and without loss of generality setting $c_3 = 0, c_2 = -1$ as is usual for the Burgers' equation, in the $\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}$-presentation, one finally has

$$u_t + (u_{x_1} + u_{x_2})u - u_{x_1x_1} - 2u_{x_1x_2} - u_{x_2x_2} = 0 \tag{74}$$

$$u_{x_1x_1} = \frac{2u_{x_1}^2}{u + c_1} + \lambda u_{x_1} \tag{75}$$

$$u_{x_2x_1} = u_{x_1}\left(\frac{u_{x_2} - u_{x_1}}{u + c_1} + \frac{u + c_1}{2} - \lambda\right). \tag{76}$$

The IMS (75), (76) when $\lambda \neq 0$ leads to the following SF (see the appendix for the calculation details):

$$u(x_1, x_2, t) = -2\left[\frac{\lambda\, e^{\lambda x_1 + \lambda(c_1 - \lambda)t} + \theta_{x_2}}{\lambda\, e^{\lambda x_1 + \lambda(c_1 - \lambda)t} + \theta}\right] - c_1 + 2\lambda \qquad \theta = \theta(x_2, t), \lambda \neq 0 \tag{77}$$

In so doing, the function $\theta(x_2, t)$ satisfies the equation

$$\theta_t - \theta_{x_2x_2} + (2\lambda - c_1)\theta_{x_2} = 0$$

and (77) has the limits

$$\lim_{x_1 \to +\infty} u(x_1, x_2, t) = -c_1 \qquad \lim_{x_1 \to -\infty} u(x_1, x_2, t) = -2\frac{\theta_{x_2}}{\theta} - c_1 + 2\lambda. \tag{78}$$

In other words, the combination (78) itself satisfies the Burgers' equation.

Analogously, for $\lambda = 0$, one obtains the SF

$$u(x_1, x_2, t) = -\frac{2(1+\theta_{x_2})}{(x_1 + c_1 t + \theta)} - c_1 \qquad \theta = \theta(x_2, t) \tag{79}$$

when $\theta(x_2, t)$ satisfies the equation

$$\theta_t - \theta_{x_2 x_2} - c_1 \theta_{x_2} = 0.$$

Here, however, one has

$$\lim_{x_1 \to \infty} u(x_1, x_2, t) = -c_1$$

and we obtain no combination of $\theta(x_2, t)$ satisfying the initial equation.

By this means, from the IMS (75), (76) we have two SFs, (77) and (79). In both cases, setting $\theta = 0$, we have the unperturbed solitonic solutions.

Example 8. Although the theory allows us to pose the problem of finding IMSs generally enough, e.g., 'find all the third-order evolution equations with all the fourth-order envelope equations', at the present moment, however, a similar problem for the second order is likely to be insoluble for a reasonable CPU time. Moreover, in doing so, on the one hand, in the framework of computer algorithms it is necessary to process and analyse the results of many branches and, on the other hand, not all of the structures found are associated with interesting solution dynamics.

In this example we will consider the following family of reaction–diffusion equations:

$$u_t = f_0(u) + f_1(u)u_x + f_2(u)u_x^2 + u_{xx} \tag{80}$$

where the f_i and u are real-value functions. Such types of modified diffusion equations appear in a number of interesting models (see, e.g., [16, 17]). Our goal here will be to determine all such equations having the simplest kink solution analogous to the Burgers' one associated with the soliton envelope equation

$$u_{x_1 x_1} = \frac{u_{x_1}}{u}\left(2u_{x_1} + \lambda u\right) \qquad \lambda \neq 0 \tag{81}$$

i.e. the same type of equation as (75), but with $\lambda \neq 0$ (here the pole case with $\lambda = 0$ leads to a nonphysical result, and $c_1 = 0$ without loss of generality).

The compatibility conditions $u_{t x_1 x_1} - u_{x_1 x_1 t} = 0$ for (80) and (81) are as follows:

$$\begin{aligned} &u_{11}^2(q_2 u_{01} + 2f_2) + 2u_{11}\left(u_{01} f_{1u} + 2u_{01}u_1 f_{2u} + u_1 q_{uu_{01}}\right) + u_{01}^2 f_{02u} - u_{01} f_{0u} q_{u_{01}} \\ &\quad + f_{0u} q + u_{01}^2 u_1 f_{12u} - u_{01}u_1 f_{1u} q_{u_{01}} + u_1 f_{1u} q + u_{01}^2 u_1^2 f_{22u} \\ &\quad - u_{01} u_1^2 f_{2u} q_{u_{01}} + u_1^2 f_{2u} q + u_1^2 q_{2u} - q_u f_0 + u_1^2 q_u f_2 = 0 \end{aligned} \tag{82}$$

so that only the simplest linkage equation is possible

$$u_{xx_1} = g\left(u, u_{x_1}, u_x\right). \tag{83}$$

(Another of the linkage equations according to proposition 4, $u_{xxx_1} = g\left(u, u_{x_1}, u_x, u_{xx_1}\right)$, appears to be incompatible with the concrete form (80).) And we have

$$\begin{aligned} &u_2^2 g_{2u_1} + u_2\left(u_{01} f_{1u} + 2u_{01}u_1 f_{2u} + 2u_1 g_{uu_1} + 2g_{u_{01}u_1} g + 2f_2 g\right) + u_{01}u_1 f_{02u} - u_{01} f_{0u} g_{u_{01}} \\ &\quad - u_1 f_{0u} g_{u_1} + f_{0u} g + u_{01}u_1^2 f_{12u} - u_{01}u_1 f_{1u} g_{u_{01}} - u_1^2 f_{1u} g_{u_1} + 2u_1 f_{1u} g \\ &\quad + u_{01}u_1^3 f_{22u} - u_{01}u_1^2 f_{2u} g_{u_{01}} - u_1^3 f_{2u} g_{u_1} + 3u_1^2 f_{2u} g \\ &\quad + 2u_1 g_{uu_{01}} g + u_1^2 g_{2u} - g_u f_0 + u_1^2 g_u f_2 + g_{2u_{01}} g^2 = 0 \end{aligned} \tag{84}$$

$$u_1 q_u + q_{u_{01}} g - u_{01} g_u - g_{u_{01}} q - g_{u_1} g = 0 \tag{85}$$

as its compatibility conditions with (80) and (81), respectively.

The equations (82), (84), (85) with g instead of u_{11} in (82) make the system for determining the functions f_0, f_1, f_2 and q, g and can be simplified by the above-mentioned computer packages. To avoid consideration of the trivial cases of (80) linearizable by point transformations (see, e.g., (59) and (61)), we will immediately introduce the inequality related to the system under consideration

$$\left(f_{02u} + f_{0u} f_2 + f_0 f_{2u}\right)^2 + f_{1u}^2 \neq 0 \tag{86}$$

before performing the calculations.

Simplification of (82), (84)–(86) gives us several different variants. A further analysis shows that only two of them may have dynamics that are interesting from the physical viewpoint, namely

$$\begin{aligned} g_{0u_{01}} &= \frac{g_0}{u_{01}} \qquad & g_{0u} &= \frac{g_0(uf_2+2)}{2u} \qquad & g_1 &= \frac{u_{01}(2-uf_2)}{2u} \\ f_0 &= 0 \qquad & f_{1u} &= -\frac{g_0(uf_2+2)}{uu_{01}} \qquad & f_{2u} &= \frac{f_2^2}{2} \end{aligned} \tag{87}$$

and

$$\begin{aligned} & g_{0u} = 0 \qquad g_{0u_{01}} = \frac{g_0}{u_{01}} \qquad g_1 = 2\frac{u_{01}}{u} \\ & f_{02u} = \frac{2}{u^2 u_{01}^2}\left(uu_{01}^2 f_{0u} + u^2 f_2 g_0^2 + 2ug_0^2 - u_{01}^2 f_0\right) \\ & f_{1u} = -2\frac{g_0(uf_2+2)}{uu_{01}} \qquad f_{2u} = -2\frac{(1+uf_2)}{u^2} \end{aligned} \tag{88}$$

with

$$g(u, u_{01}, u_1) = g_0(u, u_{01}) + g_1(u, u_{01})u_1. \tag{89}$$

Proceeding in the same way as in the previous example, the forms of $f_i(u)$ $(i = \overline{0,2})$ in (80) and the related IMSs and SFs are easily found. Below the final results are presented.

The system (87) with (89) in the general case (it also contains the Burgers case as a degenerate example) leads to the following expression:

$$u_t = \left(\frac{2c_1c_2}{c_1 - u} + c_3\right)u_x + \left(\frac{2}{c_1 - u}\right)u_x^2 + u_{xx} \qquad c_1, c_2, c_3 = \text{const} \quad c_1c_2 \neq 0 \tag{90}$$

(at $c_1c_2 = 0$, according to (86), it is linearizable by a point transformation) with the IMS (81), (83) in the D_x-presentation

$$\begin{aligned} u_{x_1x_1} &= \frac{u_{x_1}}{u}\left(2u_{x_1} + \lambda u\right) \qquad \lambda \neq 0 \\ u_{xx_1} &= \frac{u_{x_1}}{u - c_1}\left[c_2 u + (2u - c_1)\frac{u_x}{u}\right]. \end{aligned}$$

The latter corresponds to the SF

$$u(x_1, x_2, t) = \frac{A(x_2, t)}{\mathrm{e}^{\lambda[x_1 + (\lambda - 2c_2 - c_3)t] + \varphi(x_2,t)} + 1} \tag{91}$$

where the phase $\varphi(x_2, t)$ is linked by the integral relation with amplitude $A(x_2, t)$

$$\varphi(x_2, t) = \int \left(\frac{A_{x_2} + c_2 A}{A - c_1} - \lambda\right) \mathrm{d}x_2.$$

The solutions (91) have the properties (here $\lambda > 0$ for definiteness)

$$\lim_{x_1\to+\infty} u(x_1, x_2, t) = 0 \qquad \lim_{x_1\to-\infty} u(x_1, x_2, t) = A(x_2, t).$$

Simply speaking, the function A is an arbitrary solution of the original equation (90).

Solving the other system (88) brings a simpler result. The NPDE (80) is as follows:

$$u_t = c_1^2 c_2 + c_4 u + c_3 u^2 + \left(2\frac{c_1 c_2}{u} + c_5\right) u_x + \left(\frac{c_2 - 2u}{u^2}\right) u_x^2 + u_{xx}$$
$$c_i = \text{const} \quad i = \overline{1, 5} \tag{92}$$

and it is not linearizable directly if ($c_i \in \mathbb{R}$ on the condition)

$$(c_1 c_2)^2 + \left[c_2\left(c_4 - 2c_1^2\right)\right]^2 \neq 0.$$

While the IMS has the form

$$u_{x_1 x_1} = \frac{u_{x_1}}{u}(2u_{x_1} + \lambda u) \qquad \lambda \neq 0$$
$$u_{x x_1} = c_1 u_{x_1} + 2\frac{u_{x_1} u_x}{u}$$

and at $c_1 = \lambda$ corresponds to the simplest nontrivial SF

$$u(x_1, x_2, t) = \frac{1}{\mathrm{e}^{\lambda x_1 + (c_4 - c_5\lambda - \lambda^2)t} + \varphi(x_2, t)} \tag{93}$$

with the properties ($\lambda > 0$)

$$\lim_{x_1\to+\infty} u(x_1, x_2, t) = 0 \qquad \lim_{x_1\to-\infty} u(x_1, x_2, t) = \frac{1}{\varphi(x_2, t)}$$

That is $\varphi(x_2, t)^{-1}$ satisfies the initial equation (92). As seen from (93), the characteristic of the present SF is that in contrast to (91), in certain situations, the appearance of a singularity on the real axis is possible.

4. IMSs and truncated singular expansions

In the work [1] not only the multidimensional superposition principle itself was introduced, but also with its help the linkage between a presentation of solutions by truncated singular expansions [3, 4] and existence of solitons in the related equations was shown as well. In so doing, from the standpoint of the multidimensional superposition principle instead of the NPDEs themselves, we considered the equations of the system

$$V_x = -V^2 - \frac{S}{2}$$
$$V_t = CV^2 - C_x V + \frac{CS + C_{xx}}{2} \qquad V = V(x, t) \quad S = S(x, t) \quad C = C(x, t).$$

(S and C are subject to the formal compatibility condition $S_t + C_{xxx} + 2SC_x + CS_x = 0$) for the basic function V of the above-mentioned expansions

$$u(x, t) = \sum_{i=m}^{0} w_i(S, C, S_x, C_x, S_t, C_t, \ldots)V^i \qquad m \in N \tag{94}$$

The SF for V obtained in such a way together with the auxiliary expressions for C and S thereafter lead, in view of (94), to SFs for concrete NPDEs.

To be more precise, it was demonstrated that the functions V and S, C can be given in the following manner (t was also split there):

$$
\begin{aligned}
V &= \left(\frac{k+\theta_{x_2}}{2}\right)\tanh\left(\frac{kx_1+\omega t_1+\theta}{2}\right) - \frac{\theta_{x_2x_2}}{2(k+\theta_{x_2})} \qquad \theta = \theta(x_2, t_2) \\
S &= -\frac{(k+\theta_{x_2})^2}{2} - \frac{3}{2}\left(\frac{\theta_{x_2x_2}}{k+\theta_{x_2}}\right)^2 + \frac{\theta_{x_2x_2x_2}}{k+\theta_{x_2}} \\
C &= -\left(\frac{\omega+\theta_{t_2}}{k+\theta_{x_2}}\right) \qquad k, \omega = \mathrm{const} \quad k \neq 0
\end{aligned}
\tag{95}
$$

or

$$
\begin{aligned}
V &= \frac{1+\theta_{x_2}}{x_1+\omega t_1+\theta} - \frac{\theta_{x_2x_2}}{2(1+\theta_{x_2})} \qquad \theta = \theta(x_2, t_2) \\
S &= -\frac{3}{2}\left(\frac{\theta_{x_2x_2}}{1+\theta_{x_2}}\right)^2 + \frac{\theta_{x_2x_2x_2}}{1+\theta_{x_2}} \\
C &= -\left(\frac{\omega+\theta_{t_2}}{1+\theta_{x_2}}\right) \qquad \omega = \mathrm{const}
\end{aligned}
\tag{96}
$$

if the related expression (94) with $\theta = 0$ is a solution of an equation of interest, and simultaneously $\theta = 0$ satisfies its singular manifold equation. In so doing, expression (94) is the sought for SF.

Hence, in view of the form of (94), all such SFs will have the following structure:

$$
u = \sum_{i=m}^{0} \varphi_{i+1}(x_2, t)\left(\frac{\mathrm{e}^{kx_1+\varphi_0(x_2,t)}}{\mathrm{e}^{kx_1+\varphi_0(x_2,t)}+1}\right)^i \qquad k \neq 0
$$

for the case (95) and

$$
u = \sum_{i=m}^{0} \frac{\varphi_{i+1}(x_2, t)}{(x_1+\varphi_0(x_2, t))^i}
$$

for (96) respectively that determines a soliton envelope equation. The last one can be easily derived from the appropriate 'generating' equation

$$
u_{(m+1)\xi} = 0
$$

where

$$
\xi = \frac{\mathrm{e}^{kx_1+\varphi_0}}{\mathrm{e}^{kx_1+\varphi_0}+1} \qquad k \neq 0, \varphi_0 = \varphi_0(x_2, t)
$$

or

$$
\xi = \frac{1}{x_1+\varphi_0} \qquad \varphi_0 = \varphi_0(x_2, t)
$$

In particular, when $m = 1$ one has for the first case of ξ

$$
2u_{x_1x_1x_1}u_{x_1} - 3u_{x_1x_1}^2 + k^2u_{x_1}^2 = 0
$$

Setting $k = 0$, we also arrive at the equation corresponding to the second case. As a result, we can avoid the use of truncated series and consider a suitable soliton envelope equation

$$
Q(u, u_{x_1}, \ldots, u_{(m+1)x_1}) = 0
$$

instead. The latter together with an original NPDE (more precisely its d-adjoint) make an initial set of equations which can be processed by any of the existing specialized computer algebra programs.

As examples, we give the results for two well-known equations, namely, the KdV and MKdV equations.

Example 9 (The MKdV equation)**.** Since the MKdV has the truncated singular expansion with $m = 1$ [3], the initial system for $u(x_1, x_2, t)$ has the form

$$u_t - 6u^2u_x + u_{xxx} = 0 \qquad u = u(x_1, x_2, t) \qquad 2u_{x_1x_1x_1}u_{x_1} - 3u^2_{x_1x_1} + k^2u^2_{x_1} = 0 \tag{97}$$

(we will consider only the kink case and set $k > 0$ for definiteness, and without loss of generality) and it is closed by the following equations:

$$u_{x_1x} \mp 2uu_{x_1} = 0 \qquad u_{x_1x_1x} \mp 2\left(u^2_{x_1} + uu_{x_1x_1}\right) = 0$$

describing the linkages to the three ‘parameters’ associated with the soliton envelope equation (97). The related SF,

$$u(x_1, x_2, t) = \pm\left[\left(\frac{k+\theta_{x_2}}{2}\right)\tanh\left(\frac{kx_1 + \frac{k^3}{2}t + \theta}{2}\right) - \frac{\theta_{x_2x_2}}{2\left(k+\theta_{x_2}\right)}\right] \qquad \theta = \theta(x_2, t) \quad k > 0$$

with θ satisfying the equation

$$2\theta_t + 2\theta_{x_2x_2x_2} - \theta^3_{x_2} - 3k\theta^2_{x_2} - 3k^2\theta_{x_2} - \frac{3\theta^2_{x_2x_2}}{\theta_{x_2} + k} = 0 \qquad \theta = \theta(x_2, t)$$

(the calibration

$$\varphi_0(x_2, t) = \frac{k^3}{2}t + \theta(x_2, t)$$

has been used for clarity as before in example 7, appendix) corresponds to the superposition of the kink

$$\lim_{\theta\to 0} u(x_1, x_2, t) = \pm\frac{k}{2}\tanh\left(\frac{kx_1 + \frac{k^3}{2}t}{2}\right)$$

and an arbitrary perturbation

$$\lim_{x_1\to+\infty} u(x_1, x_2, t) = \pm\left[\left(\frac{k+\theta_{x_2}}{2}\right) - \frac{\theta_{x_2x_2}}{2\left(k+\theta_{x_2}\right)}\right]$$

$$\lim_{x_1\to-\infty} u(x_1, x_2, t) = \pm\left[-\left(\frac{k+\theta_{x_2}}{2}\right) - \frac{\theta_{x_2x_2}}{2\left(k+\theta_{x_2}\right)}\right].$$

This shows that in the process of their interaction the latter modulates the kink’s amplitude and phase and also leads to the appearance of an additional additive component in the solution.

Example 10 (The KdV equation)**.** Since for the KdV

$$u_t + 2uu_x + u_{xxx} = 0 \qquad u = u(x_1, x_2, t)$$

$m = 2$ [3], it requires more computational time than when $m = 1$, so we consider its potential version ($u = v_x$) instead. The initial system has the form

$$v_t + v^2_x + v_{xxx} = 0 \qquad v = v(x_1, x_2, t) \qquad 2v_{x_1x_1x_1}v_{x_1} - 3v^2_{x_1x_1} + k^2v^2_{x_1} = 0$$

(as before let $k > 0$) and is closed by the equations

$$3v_{x_1xxx}v^2_{x_1} + 3v^3_{x_1x} + 2v^3_{x_1}v_{xx} - 6v_{x_1}v_{x_1x}v_{x_1xx} = 0 \qquad 3v_{x_1x_1x}v_{x_1} + v^3_{x_1} - 3v_{x_1x}v_{x_1x_1} = 0.$$

As a result, for the KdV one has the following expression $\left(u = v_{x_1} + v_{x_2}\right)$:

$$u(x_1, x_2, t) = -\frac{3}{2}\left(k + \theta_{x_2}\right)^2 \tanh^2\left(\frac{kx_1 - k^3t + \theta}{2}\right) + 3\theta_{x_2x_2}\tanh\left(\frac{kx_1 - k^3t + \theta}{2}\right)$$
$$+ \frac{3}{4}\left(k + \theta_{x_2}\right)^2 + \frac{3}{4}\left(\frac{\theta_{x_2x_2}}{k + \theta_{x_2}}\right)^2 - \frac{3}{2}\left(\frac{\theta_{x_2x_2x_2}}{k + \theta_{x_2}}\right) + \frac{3}{4}k^2$$
$$\theta = \theta(x_2, t) \quad k > 0$$

after the calibration

$$\varphi_0(x_2, t) = -k^3t + \theta(x_2, t)$$

with $\theta(x_2, t)$ being a solution of the equation

$$2\theta_t + 2\theta_{x_2x_2x_2} - \theta_{x_2}^3 - 3k\theta_{x_2}^2 - \frac{3\theta_{x_2x_2}^2}{\theta_{x_2} + k} = 0 \qquad \theta = \theta(x_2, t).$$

Again, we see that a perturbation modulates the phase, but the general deformation of the envelope is much more complicated here. In so doing, the separated soliton and a localized $(\theta(\pm\infty, t) = \theta_{\pm\infty} = \text{const})$ perturbation will have the following form (before and after an interaction):

$$\lim_{x_2\to\pm\infty} u(x_1, x_2, t) = \frac{3}{2}k^2\left[1 - \tanh^2\left(\frac{kx_1 - k^3t + \theta_{\pm\infty}}{2}\right)\right]$$

and

$$\lim_{x_1\to\pm\infty} u(x_1, x_2, t) = \pm 3\theta_{x_2x_2} - \frac{3}{4}\left(k + \theta_{x_2}\right)^2 + \frac{3}{4}\left(\frac{\theta_{x_2x_2}}{k + \theta_{x_2}}\right)^2 - \frac{3}{2}\left(\frac{\theta_{x_2x_2x_2}}{k + \theta_{x_2}}\right) + \frac{3}{4}k^2$$

respectively.

Note 7. An IMS with the above types of soliton envelopes can be an alternative to the standard technique of the singular manifold method with a substitution of truncated expansions. Moreover, for equations possessing the Painlevé property but not admitting truncation of such expansions, IMSs can be applied as an approach for summing the related infinite Laurent series and studying their properties.

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Appendix

Integrating equation (75), the soliton envelope equation, one has

$$u(x_1, x_2, t) = -2\left[\frac{\lambda\, e^{\lambda x_1} + \varphi_1(x_2, t)}{e^{\lambda x_1} + \varphi_0(x_2, t)}\right] - c_1 + 2\lambda \qquad \text{(A1)}$$

so that the functions φ_0 and φ_1 are still undefined here. After substitution of (A1) into (76), we have the linkage between them

$$\varphi_1 = \varphi_{0x_2}$$

while after the separation of the variables x_2 and x_1 (74) leads to the two equations

$$-\varphi_{0tx_2}\varphi_0 + \varphi_{0t}\varphi_{0x_2} + \varphi_{03x_2}\varphi_0 - \varphi_{02x_2}\varphi_{0x_2} + (c_1 - 2\lambda)\left(\varphi_{02x_2}\varphi_0 - \varphi_{0x_2}^{2}\right) = 0$$

$$-\varphi_{0tx_2} + \lambda\varphi_{0t} + \varphi_{03x_2} + (c_1 - 3\lambda)\varphi_{02x_2} + \lambda(3\lambda - 2c_1)\varphi_{0x_2} + \lambda^2(c_1 - \lambda)\varphi_0 = 0$$

which reduce to the single equation for φ_0

$$\varphi_{0t} = \varphi_{02x_2} + (c_1 - 2\lambda)\varphi_{0x_2} + \lambda(\lambda - c_1)\varphi_0 \qquad \varphi_0 = \varphi_0(x_2, t)$$

Next, introducing another function according to the relation

$$\varphi_0(x_2, t) = e^{\lambda(\lambda - c_1)t}\theta(x_2, t)$$

the new function $\theta(x_2, t)$ will already satisfy the equation

$$\theta_t = \theta_{x_2x_2} + (c_1 - 2\lambda)\theta_{x_2} \qquad \theta = \theta(x_2, t)$$

possessing the trivial solution $\theta = 0$.

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